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APPROXIMATE METHOD FOR THE DETERMINATION OF THE NATURAL OSCILLATION FREQUENCIES OF CYLINDRICAL, CONIC AND TOROIDAL SHELLS

I. I. Meyerovich

ABSTRACT

Development of a method for the approximate determination of the flexural vibrations of shells of revolution, where the vibrations involve the appearance of nodal lines along the generatrix and in the meridional directions. The method is based on the use of a Ritz-type strain-energy approach and the selection of the approximating functions such that--besides the boundary conditions--they satisfy the additional conditions of the absence of fluctuations of circumferential strains and stresses and the absence of shear in the middle surface. A detailed analysis is given of the frequency spectrum of a cylindrical shell with arbitrary boundary conditions and of the combined oscillations of two cylindrical shells of different rigidity. These, and the results obtained for a closed thin-walled toroidal shell suspended in free space, are compared with the experiment.

Introduction

The present work considers the flexural oscillations of shells of revolution which exhibit nodal lines in meridional directions and along the generatrix. /148*

A series of Soviet works have been devoted to the investigation of this class of oscillations -- the works of Breslavskiy, ref. 3, Filippov, ref. 7, -- and also works conducted abroad and Arnold Warburton, ref. 4, which pertain primarily to cylindrical shells and to conic shells with small conicity.

We assume that no limitations are imposed on the displacement along the normals to the middle surface w, along the tangents to the circular cross

*Numbers given in margin indicate pagination in original foreign text.

sections v and along the shell u. We then obtain a characteristic equation of the third order for computing the square of the frequency of natural oscillations associated with a cylindrical shell which has a hinge support and a fixed number of waves: $p^6 - L_2p^4 + L_1p^2 - L_0 = 0$. The roots of this equation differ radically from each other.

The following expressions are used as approximation equations for determining the square of the lowest frequency

$$p^2 = \frac{L_0}{L_1}, \tag{3}$$

$$p^2 = \frac{L_0}{L_1}\left(1 + \frac{L_2 L_0}{L_1^2}\right), \tag{4}$$

which agree well with experimental results, if the number of waves in the circumferential direction is $n \geq 4$.

If we neglect the coupling between the forms of oscillations containing a different number of waves in the longitudinal direction, these equations may also be utilized for other boundary conditions.

From the analysis of the amplitude of oscillations it follows that the lowest frequency is determined primarily by the flexural oscillations w, and the second by longitudinal displacements u, whereas the third is determined by shear oscillations v.

For a single shell Flyugge (reference 6) presents the following frequency values

$$p_1 = 18.36 \text{ Hz}, \; p_2 = 918 \text{ Hz}, \; p_3 = 1453 \text{ Hz}.$$

The ratios of amplitudes are as follows:

for p_1w: u: v = 1: -0.25: -0.04,

for p_2u: v: w = 1: 0.076: 0.0526,

for p_3v: u: w = 1: -0.146: 0.245.

Since, in practice, in order to compute p_1 it is not necessary to know p_2 and p_3, we can impose specific relationships on u, v and w, and in place of the cubic equation for p^2 we can consider an equation of the form $A_1p^2 + A_2 = 0$.

We solve this problem by using the Ritz energy method, selecting the approximation functions in such a way that in addition to boundary conditions they satisfy auxiliary conditions stipulating that there is an absence of oscillations producing elongations in the ring direction and an absence of shear in the middle surface.

In order to convince ourselves that the selected hypotheses are expedient and that it is possible to apply them to a more complex conical or toroidal configuration involving shells with various boundary conditions, we carried out a detailed analysis in the first part of the work pertaining to the frequency spectrum of cylindrical shells with arbitary boundary conditions. We also considered the combined oscillations of two cylindrical shells of different rigidity.

In the second part we considered the oscillations of a conical shell hinged along the edges.

The theoretical results have been verified experimentally.

We note that there are no approximate equations available for computing the natural oscillations of conical shells.

In the third part we considered the oscillations of a closed, thin-walled toroidal shell in free space. An exact solution of this problem has not been obtained to date. There are also no data on the experimental investigations of such shells.

I. Cylindrical Shells

1. Basic Propositions

In order to analyze the spectrum of frequencies associated with the natural osciallations of the shell, we designate the displacements normal to the coordinate line x = const, lying on the middle surface, by w, the displacements tangential to the circular sections by v and the longitudinal displacements by u (figure 1a).

Furthermore, by introducing a curvilinear system of coordinates x, θ, we assume that u, v and w may be expressed as a sum of the products of two functions, of which one depends on x and the other on θ. The number of waves in the circumferential direction for a closed shell must be even and can be represented as a function of cos $n\theta$ or sin $n\theta$.

We write the displacements in the following form

$$\left.\begin{aligned} w&=\sum\sum A_{mn}W_m(x)\cos n\theta,\\ v&=\sum\sum B_{mn}V_m(x)\sin n\theta,\\ u&=\sum\sum C_{mn}U_m(x)\cos n\theta.\end{aligned}\right\} \qquad (1.1)$$

The strain parameters of the cylindrical shell are determined by the following expressions /150

$$\left.\begin{aligned}
&\varepsilon_1=\frac{\partial u}{\partial x}, \quad \varepsilon_2=\frac{\partial v}{R\,\partial\theta}-\frac{w}{R}, \quad \omega=\frac{\partial v}{\partial x}+\frac{\partial u}{R\,\partial\theta},\\
&\varkappa_1=\frac{\partial^2 w}{\partial x^2}, \quad \varkappa_2=\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial v}{\partial\theta}\right), \quad \tau=\frac{\partial}{\partial x}\left(\frac{\partial w}{R\partial\theta}+\frac{v}{R}\right).
\end{aligned}\right\} \qquad (1.2)$$

We assume that the shell is not extended in the transverse direction and that there is no shear in the middle-surface, i.e., $\varepsilon_2 = \omega = 0$

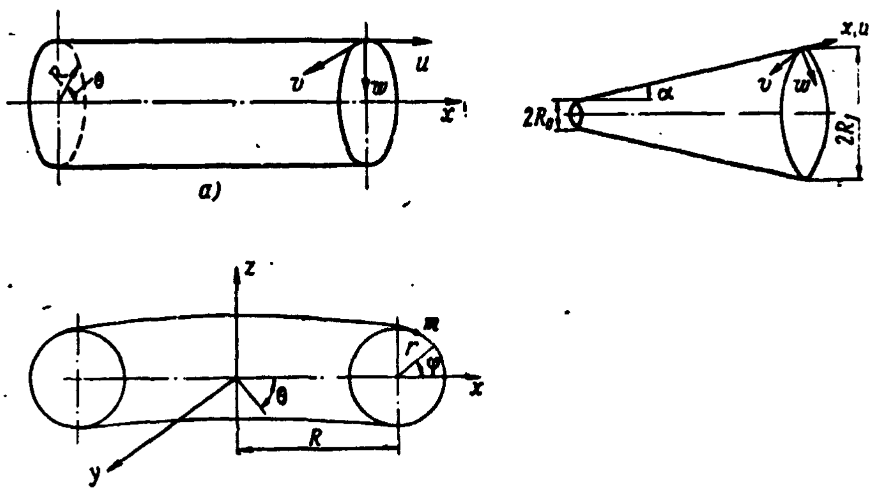

Figure 1. a, Cylindrical shell; b, conical shell; c, toroidal shell.

Then it is obviously possible to have a relationship between the functions w, v and u. Specifically, for each fixed n the following equalities must be satisfied

$$\left.\begin{aligned}
&B_{mn}V_m(x)\,n-A_{mn}W_m(x)=0,\\
&B_{mn}V'_m(x)-C_{mn}\frac{U_m(x)}{R}\,n=0.
\end{aligned}\right\} \qquad (1.3)$$

For any value of x we obtain the following from equality (1.3.)

$$\left.\begin{aligned}
&B_{mn}V_m(x)=\frac{A_{mn}}{n}\,W_m(x),\\
&C_{mn}U_m(x)=\frac{B_{mn}}{n}\,V'_m(x)\,l=\frac{A_{mn}}{n^2}\,W'_m(x)\,R.
\end{aligned}\right\} \qquad (1.4)$$

Substituting (1.4) into (1.1) and (1.2), we find an expression for displacements and deformations

$$\left.\begin{aligned} w&=\sum\sum A_{mn}W_m(x)\cos n\theta\\ v&=\sum\sum A_{mn}\frac{W_m(x)}{n}\sin n\theta,\\ u&=\sum\sum A_{mn}\frac{W'_m(x)}{n^2}R\cos n\theta \end{aligned}\right\} \tag{1.5}$$

and

$$\left.\begin{aligned} \varepsilon_1&=\sum\sum\frac{A_{mn}}{n^2}W''_m(x)R\cos n\theta,\\ \varepsilon_2&=\omega=0,\\ \varkappa_1&=\sum\sum A_{mn}W''_m(x)\cdot\cos n\theta,\\ \varkappa_2&=\sum\sum\frac{A_{mn}}{R^2}W_m(x)(1-n^2)\cos n\theta,\\ \tau&=\sum\sum\frac{A_{mn}}{R}W'_m(x)\frac{1-n^2}{n}\sin n\theta. \end{aligned}\right\} \tag{1.6}$$

We determine the natural frequencies in the form of the shell oscillations with the Ritz method. For the approximation function $W_m(x)$ we select the natural function of oscillations of an elementary beam element cut out along the generatrix. W_m satisfies the equation /151

$$EJ\frac{d^4W}{dx^4}-p^2\varrho hW=0,$$

from which it follows that

$$W_m(x)=C_1\operatorname{ch}k_m\bar{x}+C_2\operatorname{sh}k_m\bar{x}+C_3\cos k_m\bar{x}+C_4\sin k_m\bar{x}. \tag{1.7}$$

To compute the parameters A_{mn} and the natural frequency of the shell we use the following system of equations

$$\sum\sum\left(\frac{\partial}{\partial A_{mn}}-p^2\frac{\partial T}{\partial A_{mn}}\right)=0, \tag{1.8}$$

where P is the potential energy of the shell;

T is the kinetic energy of the shell which undergoes oscillations with a frequency p.

Under the adopted assumption.

$$P = \frac{Eh}{1-\mu^2}\int_0^l\int_0^{2\pi} \varepsilon_1^2 R dx d\theta + \frac{Eh^3}{12(1-\mu^2)}\int_0^l\int_0^{2\pi}[\varkappa_1^2+\varkappa_2^2+2\mu\varkappa_1\varkappa_2+2\times$$

$$\times(1-\mu)\tau^2]Rdxd\theta, \qquad (1.9)$$

$$T=\varrho h\int_0^l\int_0^{2\pi}[w_m^2+v_m^2+u_m^2]\,Rdxd\theta.$$

Differentiating (1.8) and (1.9) with respect to A_{mn} and substituting into (1.7), we obtain the following system of equations for each fixed value of n

$$A_{1n}(a_{1n}^{1n}-p^2b_{1n}^{1n})+A_{2n}(a_{2n}^{1n}-p^2b_{2n}^{1n})+\ldots+A_{mn}(a_{mn}^{1n}-p^2b_{mn}^{1n})=0,$$

$$A_{1n}(a_{1n}^{2n}-p^2b_{1n}^{2n})+A_{2n}(a_{2n}^{2n}-p^2b_{2n}^{2n})+\ldots+A_{mn}(a_{mn}^{2n}-p^2b_{mn}^{2n})=0,$$

$$\cdots\cdots\cdots\cdots \qquad (1.10)$$

$$A_{1n}(a_{1n}^{mn}-p^2b_{1n}^{mn})+A_{2n}(a_{2n}^{mn}-p^2b_{2n}^{mn})+\ldots+A_{mn}(a_{mn}^{mn}-p^2b_{mn}^{mn})=0.$$

The coefficient A_{mn}^{mn} and A_{mn}^{pm} after intergration over φ will be /152

$$a_{mn}^{mn}=\frac{Eh}{1-\mu^2}\pi R\left\{\int_0^l\frac{W_m''^2}{n^4}R^2dx+\frac{h^2}{12}\int_0^l[W_m''^2+\right.$$

$$+\frac{W_m^2}{R^4}(1-n^2)^2+2\mu\frac{1-n^2}{R^2}W_m''W_m+ \qquad (1.11)$$

$$\left.+2(1-\mu)\frac{W_m'^2}{R^2}\frac{(1-n^2)^2}{n^2}\Big]\right\}dx,$$

$$a_{mn}^{pn}=\frac{Eh^3}{12(1-\mu^2)}\pi R\int_0^l\Big[\frac{\mu}{R^2}(1-n^2)(W_m''W_p+$$

$$+W_p''W_m)+2(1-\mu)\frac{1-n^2}{n^2}\frac{W_m'W_p'}{R^2}\Big]dx, \qquad (1.12)$$

$$\left.\begin{aligned} b_{mn}^{mn}&=\pi\varrho hR\int_0^l\Big[W_m^2\Big(1+\frac{1}{n^2}\Big)+\frac{W_m'^2R^2}{n^4}\Big]d\bar{x}\\ b_{mn}^{pn}&=\pi\varrho hR\int_0^l\frac{W_m'W_p'}{n^4}dx.\end{aligned}\right\} \qquad (1.13)$$

In deriving equations (1.12) and (1.13) we took into account the orthogonality of the beam functions, i.e.,

$$\left.\begin{aligned}\int_0^l W_m(x)W_p(x)\,dx=0,\\ \int_0^l W''_m(x)W''_p(x)\,dx=0.\end{aligned}\right\} \quad (1.14)$$

Having considered the general method of solving the problem of determining the natural oscillations of the shell, we proceed with the consideration of specific cases.

2. A Hinge-supported Shell

A large number of theoretical and experimental works have been devoted to the investigation of oscillations of a shell of this type. Therefore, we shall begin our investigations with this case, which is most simple from the standpoint of the theoretical solution.

Indeed, if the shell is hinged at the ends (x = 0 and x = 1), the flexure at these points becomes equal to zero and the flexural moment of the shell and the beam function will be

$$W_m(x)=\sin\frac{m\pi x}{l},\quad m=1, 2, 3\ldots \quad (2.1)$$

It follows from (1.14) that in equation (1.10) all of the supplementary coefficients become equal to zero; for each value of m we obtain the following equation for computing p^2_m /153

$$p^2_m=\frac{a^{mn}_{mn}}{b^{mn}_{mn}}, \quad (2.2)$$

which can be expanded in the form

$$p^2_m=\frac{E}{(1-\mu^2)\varrho R^2}\,\frac{k^4_m\eta^4+\beta n^4\left\{k^4_m\eta^4+(1-n^2)^2+2k^2_m\eta^2\left[\mu(n^2-1)+(1-\mu)\frac{(1-n^2)^2}{n^2}\right]\right\}}{n^4+n^2+k^2_m\eta^2}, \quad (2.3)$$

where

$$k_ml=m\pi,\quad \beta=\frac{h^2}{12R^2},\quad \eta=\frac{R}{l}.$$

In order to convince ourselves that the adopted hypotheses are expedient, we use equation (2.3) to compute five shells (the geometric characteristics are given in table 1) and to compare the resulting quantities with the computed and experimental data given by Breslavskiy (ref. 3).

TABLE 1. GEOMETRIC CHAR'CTERISTICS OF THE SHELL.

No. of shell	l мм	R мм	h мм	$\beta = \frac{h^2}{12R^2}$	$\frac{R}{l}$	$\pi\frac{R}{l}$	$\frac{1}{2\pi R}\sqrt{\frac{E}{\varrho(1-\mu^2)}}$	Remarks
1	540	125	0.8	$3{,}42\cdot10^{-6}$	0,232	0,726	6750	$E=2\cdot2\cdot10^6 \frac{\text{dan}}{\text{см}^2}$ $\mu=0.3$
2	540	175	0.8	$1{,}74\cdot10^{-6}$	0,324	1,02	4820	
3	540	200	0.8	$1{,}335\cdot10^{-6}$	0,370	1,16	4240	
4	817	200	1,4	$4{,}07\cdot10^{-6}$	0,245	0.770	4220	
5	380	200	0,8	$1{,}335\cdot10^{-6}$	0.526	1.650	4220	

TABLE 2.

Shells:	№ 1			№ 2			№ 3			№ 4			№ 5		
n	$p_э$	p_B	$p_н$	$p_э$	p_B	$p_н$	$p_э$	p_B	$p_н$	$p_э$	p_B	$p_н$	$p_э$	p_B	$p_н$
2	—	—	—	—	—	—	—	—	—	408	462	550	—	—	—
4	256	272	280	—	—	—	—	—	—	192	191	198	538	628	684
5	—	—	—	290	248	254	250	237	259	—	—	—	484	431	480
6	438	426	442	—	—	—	233	225	219	323	309	303	350	345	370
8	—	—	—	466	391	411	334	307	312	—	—	—	345	368	361
10	—	—	—	705	635	640	497	476	485	—	—	—	484	501	500
12	—	—	—	—	—	—	715	679	702	—	—	—	—	—	—
	—	—	—	—	—	—	968	924	980	—	—	—	—	—	—

B = Breslavskiy's formula

э = experimental data

н = our data

Table 2 shows the experimental values for the natural frequencies of oscillation in Hz and theoretical values computed by means of our equations and by means of the Breslavskiy equations.

Comparing the results of computing the natural frequencies by /154 means of equation (2.3) with the corresponding values of experimental frequencies computed by means of the Breslavskiy equation, we can see that in all those cases when the energy of extension can be neglected compared with the flexural energy, equation (2.3) gives results which are quite satisfactory. The order of the error in equation (2.3) and in the Breslavskiy method is the same, in spite of the fact that our equations are based on hypotheses which differ from those usually adopted and which simplify the calculations substantially.

3. Cantilever Shell

In this case the beam function $W_m(x)$ will be determined as

$$W_m(x)=\frac{1}{\sqrt{l}}[\mathrm{ch}-k_m\bar{x}-\cos k_m\bar{x}+B_m(\sin x_m\bar{x}-\mathrm{sh}\,k_m x)], \tag{3.1}$$

where

$$B_m=\frac{\mathrm{sh}\,k_m-\sin k_m}{\mathrm{ch}\,k_m+\cos k_m},\quad \bar{x}=\frac{x}{l},$$
$$k_m l=1.875,\quad 4.694,\quad 7.854\ldots \tag{3.2}$$

Substituting (3.1) into equations (1.11)-(1.12), we obtain

$$a_{mn}^{mn}=\frac{Eh}{(1-\mu^2)R^2}\pi R\left\{\frac{k_{mm}^4\eta^4}{n^4}+\beta\left[k_{mm}^4\eta^4+(1-n^2)^2+\right.\right.$$
$$\left.\left.+2\mu(1-n^2)\omega_{mm}\eta^2+2(1-\mu)\frac{(1-n^2)^2}{n^2}\alpha_{mm}\eta^2\right]\right\},$$
$$a_{mn}^{pn}=\frac{Eh\pi R}{(1-\mu^2)R^2}\beta\left[\mu(1-n^2)(\omega_{mp}+\omega_{pm})\eta^2+\right.$$
$$\left.+2(1-\mu)\alpha_{mp}\frac{(1-n^2)^2}{n^2}\eta^2\right], \tag{3.3}$$

where

$$\omega_{mp}=\int_0^l W_m'' W_p\,dx,\quad \alpha_{mp}=\int_0^l W_m' W_p'\,dx, \tag{3.4}$$
$$\eta=\frac{R}{l},\quad \beta=\frac{h^2}{12R^2}.$$

Substituting (3.1) into (1.13), we find

$$b_{mn}^{mn}=\varrho h\pi R\left[1+\frac{1}{n^2}+\alpha_{mm}\frac{\eta^2}{n^4}\right],$$
$$b_{mn}^{pn}=\varrho h\pi R\alpha_{mp}\frac{\eta^2}{n^4}. \tag{3.5}$$

We have the following table of values ω_{mp} and α_{mp} for the first three values of m (table 3).

TABLE 3.

	ω_{mp}			α_{mp}		
m \ p	1	2	3	1	2	3
1	0.880	1.881	1.57	4.69	−7.466	4.01
2	−11.66	−13.29	3.16	—	32.40	−22.27
3	27.06	−0.97	−46	—	—	77.38

By using table 3 we can write the coefficient A_{mn}^{mn} for m = 1, 2 and 3 /155 in the following manner, if we let $R/l=\eta$ and eliminate the constant π

$$\frac{(1-\mu^2)R^2}{Eh}n^4a_{1n}^{1n}=12.4\eta^4+\beta n^4\Big\{12.4\eta^4+(1-n^2)^2+$$

$$+2\eta^2\Big[-0.264(1-n^2)+3.29\frac{(1-n^2)^2}{n^2}\Big]\Big\}$$

$$\frac{(1-\mu^2)R^2}{Eh}n^4a_{2n}^{2n}=484\eta^4+\beta n^4\Big\{484\eta^4+(1-n^2)^2+ \qquad (3.6)$$

$$+2\eta^2\Big[4(1-n^2)^2+22.6\frac{(1-n^2)^2}{n^2}\Big]\Big\},$$

$$\frac{(1-\mu^2)R^2}{Eh}n^2a_{3n}^{3n}=3800\eta^4+\beta n^4\Big\{3800\eta^4+(1-n^2)^2+$$

$$+2\eta^2\Big[13.8(1-n^2)+34\frac{(1-n^2)^2}{n^2}\Big]\Big\},$$

$$n^4b_{1n}^{1n}=n^4+n^2+4.69\eta^2, \qquad (3.7)$$

$$n^4b_{2n}^{2n}=n^4+n^2+32.04\eta^2,$$

$$n^4b_{3n}^{3n}=n^4+n^2+77.38\eta^2,$$

We can make use of equation (2.2) to compute the frequencies of natural oscillations, i.e., instead of n systems from m equations we can consider only one equation consisting of the diagonal member of the matrix. The coupling which exists between forms $W_{m,n}$ and $W_{m+1,n}$ is so small that it can be neglected.

As an example we consider the spectrum of frequencies for shell number 3 see (table 1), which confirms there is an absence of coupling between modes W_{1n} and W_{2n}. For comparison purposes table 4 shows the spectrum of frequencies in Hz for hinged shells, cantilever shells and rigidly fixed shells as well as for free shells.

TABLE 4. EFFECT OF THE NATURE OF SHELL SUPPORT ON THE SPECTRUM OF NATURAL OSCILLATION FREQUENCIES.

1. Support \ n	5	6	7	10	12	14
2. Hinged	259	219	312	485	702	980
3. Cantilever	143	176	304	485	702	980
4. Rigid	525	388	378	515	721	990
5. Free	111	172	297	485	700	980

It follows from table 4 that the frequencies of natural oscillations for a cantilever shell are below the corresponding frequencies for a hinged shell when the values of n are small. As n increases and when it is permissible to neglect the effect of the potential energy of extension, the natural frequencies of oscillations no longer depend on the manner in which the shell is supported.

4. A Shell Which is Rigidly Fixed Along the Ends

In this case the function $W_m(x)$ represents a natural oscillation /156
function for a beam rigidly fixed along the edges

$$W_m(x) = \frac{1}{\sqrt{l}}\,[\sin k_m\bar{x} - \mathrm{sh}\,k_m x + B_m(\cos k_m x - \mathrm{ch}\,k_m\bar{x})], \tag{4.1}$$

where

$$B_m = -\frac{\mathrm{ch}\,k_m - \cos k_m}{\mathrm{ch}\,k_m + \sin k_m}, \tag{4.2}$$

here

$$\bar{x} = \frac{x}{l}, \quad k_m = 4.73;\ 7.853; \ldots \frac{(2m+1)\eta}{2}. \tag{4.3}$$

The value k_m is the root of the characteristic equation $\mathrm{ch}\,k_m \cos k_m = 1$. The coefficients (3.4) α_{mm} and $\omega_{,mm}$ can be written in the following manner according to (4.1) and (4.2)

$$\alpha_{mm} = \frac{k_{mm}^2}{l^2}\left(1 + \frac{2B_m}{k_m}\right); \quad \omega_{mm} = -\frac{k_m^2}{l^2}\left(1 + \frac{2B_m}{k_m}\right). \tag{4.4}$$

The coefficients of the characteristic equations (1.10) have the form

$$\begin{gathered}
n^4\frac{1-\mu^2}{En\pi}R^2 a_{mn}^{mn} = k_m^4\eta^4 + \beta n^4\Big\{k_m^4\eta^4 + (1-n^2)^2 + \\
+ 2k_m^2\eta^2\left(1 + \frac{2B_m}{k_m}\right)\left[-\mu(1-n^2) + (1-\mu)\frac{(1-n^2)^2}{n^2}\right]\Big\}, \\
n^4\frac{1-\mu^2}{Eh\pi}R^2 a_{mn}^{pn} = \beta n^4\eta^2[-\mu(1-n^2)(\omega_{mp} + \omega_{pm}) + \\
+ 2(1-\mu)\alpha_{mp}\frac{(1-n^2)^2}{n^2}\Big], \\
\frac{n^4}{\pi} b_{mn}^{mn} = n^4 + n^2 + k_m^2\eta^2\left(1 + \frac{2B_m}{k_m}\right) \\
\frac{n^4}{\pi} b_{mn}^{pn} = \alpha_{mp}\eta^2.
\end{gathered} \tag{4.5}$$

In spite of the presence of supplementary terms the coupling between the functions $W_{mn}(x)$, $W_{m+1,\,n}(x)$ is so small that in practice we can use expression (2.2) to compute the frequencies i.e.,

$$p_m^2 = A^2 \frac{k_m^4\eta^4 + \beta n^4 \left\{k_m^4\eta^4 + (1-n^2)^2 + 2k_m^2\eta^2\left(1+\frac{2B_m}{k_m}\right) \times \times\left[-\mu(1-n^2)+(1-\mu)\frac{(1-n^2)^2}{n^2}\right]\right\}}{n^4+n^2+k_m^2\eta^2\left(1+\frac{2B_m}{k_m}\right)}, \qquad (4.6)$$

$$A^2 = \frac{E}{(1-\mu^2)R^2}.$$

As a numerical example we consider the spectrum of natural frequencies of shell No. 3 (table 1).

The results obtained are shown in table 4. As was to be expected in this case, the values of the natural frequencies are higher than in the first two cases, and it is only when n = 14 that the natural frequencies cease to depend on the method used to support the shell.

5. Free Shell

A floating or free shell is a shell such that M = Q = T = 0 at both ends. /157

The bean function is written in the following form

$$W_m(x) = \frac{1}{\sqrt{l}}\left[\operatorname{sh} k_m\bar{x} + \sin k_m\bar{x} - B_m(\operatorname{ch} k_m x + \cos k_m\bar{x})\right], \qquad (5.1)$$

where

$$B_m = \frac{\operatorname{ch} k_m - \cos k_m}{\operatorname{ch} k_m + \sin k_m}, \qquad (5.2)$$

$$k_1 = 0; \quad k_2 = 4.73; \quad k_3 l = 7.85. \qquad (5.3)$$

The characteristic peculiarity of such a shell is the occurrence of transverse waves and the absence of longitudinal waves.

The general approximation equation will always be (4.6), specifically

$$p_1^2 = A^2 \frac{k_m^4\eta^4 + \beta n^4\left\{k_m^4\eta^4 + (1-n^2)^2 + 2k_m\eta^2\left[-\mu(1-n^2) \times \times\left(1+\frac{2B_m}{k_m}\right) + (1-\mu)\frac{(1-n^2)^2}{n^2}\left(1+\frac{6B_m}{k_m}\right)\right]\right\}}{n^4+n^2+k_m^2\eta^2\left(1+6\frac{B_m}{k_m}\right)}. \qquad (5.4)$$

For $k_m = 0$, equation (5.4) is transformed into the well known equation for a ring

$$p^2 = A^2 \frac{\beta n^4 (1-n^2)^2}{n^4+n^2};$$

the spectrum of frequencies in Hz is shown in table 4.

6. Combined Oscillations of Two Cylindrical Shells

Let us consider the oscillations of two cylindrical shells of the same radius R, which are rigidly attached to each other along the section $x = l_1$ (figure 2). We shall assume that shell ends $x = 0$ and $x = l_2$ are arbitrarily supported. We shall solve the problem by the energy method by assuming that the natural oscillations of the shell can be approximated by series of the following form

$$\begin{aligned} w(x) &= \sum\sum A_{mn} W_m(x) \cos n\theta, \\ v(x) &= \sum\sum A_{mn} \frac{W_m(x)}{n} \sin n\theta, \\ u(x) &= \sum\sum A_{mn} \frac{W'_m(x)}{n^2} R \cos n\theta. \end{aligned} \tag{6.1}$$

In regard to the functions $W_m(x)$, $V_m(x)$ and $U_m(x)$, they may be selected in such a way that the conditions under which the shells are joined along the boundary $x = l_1$ are satisfied, i.e.,

$$\begin{aligned} W_{m1}(l_1) &= W_{m2}(l_1), \\ V_{m1}(l_1) &= V_{m2}(l_1), \\ U_{m1}(l_1) &= U_{m2}(l_1), \\ W'_{m1}(l_1) &= W'_{m2}(l_1), \\ M_{m1}(l_1) &= M_{m2}(l_1), \\ Q_{m1}(l_1) &= Q_{m2}(l_1) \end{aligned} \tag{6.2}$$

The subscript "1" refers to the shell $0 \le x \le l_1$, and the subscript "2" refers to the shell $l_1 \le x \le l_2$. /158

Let us assume that the function $W_m(x)$ represents a beam function of the composite beam whose rigidity along the region 0-l_1 will be EJ_1, and along the region l_1-l_2 it will be EJ_2.

If at the boundary where the two shells are joined we have $W_1 = W_2$,

$$W'_1 = W'_2, \quad EJ_1 W''_1 = EJ_2 W''_2,$$
$$EJ_1 W'''_1 = EJ_2 W'''_2,$$

then condition (6.2) is satisfied automatically for V and U as a result of equation (1.5) and is approximately satisfied for M and Q.

If $W_{m1}(x)$ is the beam function for the first shell of the form (1.7) whose 2 constants are determined from conditions at the end $x = 0$, and $W_{m2}(x)$ is a beam function for the second beam whose two constants are determined from conditions at the end $x = l_2$, then in place of equations (6.2) we can write the following system of equations

$$\begin{aligned} C_1^{I} W_{m1}^{I} + C_2^{I} W_{m2}^{I} &= C_1^{II} W_{m1}^{II} + C_2^{II} W_{m2}^{II}, \\ C_1^{I} W_{m1}'^{I} + C_2^{I} W_{m2}'^{I} &= C_1^{II} W_{m1}'^{II} + C_2^{II} W_{m2}'^{II}, \\ J_1 \{C_1^{I} M_{m1}^{I} + C_2^{I} M_{m2}^{I}\} &= J_2 [C_1^{II} M_{m1}^{II} + C_2^{II} M_{m2}^{II}], \\ J_1 [C_1^{I} Q_{m1}^{I} + C_2^{I} Q_{m2}^{I}] &= J_2 [C_2^{II} Q_{m1}^{II} + C_2^{II} Q_{m2}^{II}], \end{aligned} \tag{6.3}$$

where W_i, W'_i, M_i, Q_i represent a definite grouping of hyperbolic and trigonometric functions which satisfy the conditions at the ends $x = 0$ for the first shell and $x = l_2$ for the second shell. The superscript "I" refers to the first shell, and the superscript "II" refers to the second.

The characteristic determinant of system (6.3) is written in the form

$$\begin{vmatrix} W_{m1}^{I} & W_{m2}^{I} & W_{m1}^{II} & W_{m2}^{II} \\ W_{m1}'^{I} & W_{m2}'^{I} & W_{m1}'^{II} & W_{m2}'^{II} \\ M_{m1}^{I} & M_{m2}^{I} & \nu M_{m1}^{II} & \nu M_{m2}^{II} \\ Q_{m1}^{I} & Q_{m2}^{I} & \nu Q_{m1}^{II} & \nu Q_{m2}^{II} \end{vmatrix} = 0 \tag{6.4}$$

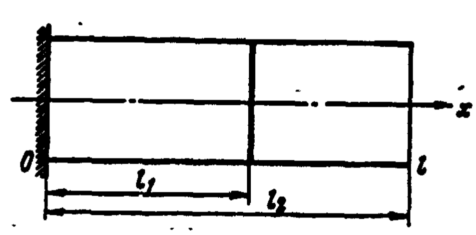

Figure 2. Composite shell.

In the expanded form equation (6.4) when $\nu = \frac{J_2}{J_2}$ (sic) has the form

$$\begin{aligned}\nu^2\Delta(M,\ Q)_2\Delta(W,\ W')_1+\nu[-\Delta(W,\ M)_1\Delta(W',\ Q)_{11}+\\ +\Delta(W',\ M)_{11}\Delta(W,\ Q)_1+\Delta(W,\ Q)_{11}\Delta(W',\ M)_1+\\ +\Delta(M,\ W)_{11}\Delta(W',\ Q)_1]+\Delta(W,\ W')_1\Delta(MQ)_2=0.\end{aligned} \tag{6.5}$$

In equation (6.5) Δ represents the elementary determinant of the following form

$$\Delta(WW')_1=\begin{vmatrix} W^{I}_{m1} & W^{I}_{m2} \\ W'^{I}_{m1} & W'^{I}_{m2}\end{vmatrix}. \tag{6.6}$$

Determinants of the form (6.6) may be expressed in terms of the Prager functions (ref. 6), i.e., /159

$$\begin{aligned}
A(k_m)&=\operatorname{ch}k_m\sin k_m+\operatorname{sh}k_m\cos k_m,\\
B(k_m)&=\operatorname{ch}k_m\sin k_m-\operatorname{sh}k_m\sin k_m,\\
S(k_m)&=2\operatorname{sh}k_m\sin k_m,\\
C(k_m)&=2\operatorname{ch}k_m\cos k_m,\\
E(k_m)&=\operatorname{ch}k_m\cos k_m+1,\\
D(k_m)&=\operatorname{ch}k_m\cos k_m-1.
\end{aligned} \tag{6.7}$$

Functions (6.7) are tabulated [1], and therefore it is not very difficult to find the roots of equations (6.5). If the rigidity of one of the shells is substantially greater than the rigidity of another shell, the dominating terms of the equation will be the coefficients in front of ν^2 or ν^0. Consequently in place of equation (5.6) we can consider the expression

$$\Delta(M,\ Q)\Delta(W,\ W')=0. \tag{6.8}$$

Equation (6.8) in turn breaks down into the following two

$$\Delta(M,\ Q)=0 \quad \text{or} \quad \Delta(W,\ W')=0,$$

i.e., at the section $x = l_1$ one of the shells may be looked upon as a rigidly supported shell for which $\Delta(W, W') = 0$, and the other may be looked upon as a freely supported shell for which $\Delta(M, Q) = 0$.

Let us rewrite (6.5) in terms of the Prager functions, if the first shell (at the section $x = 0$) is supported in a cantilever fashion and the second shell (at the section $x = l_2$) is freely supported.

For the first shell we have

$$W_1=C_1(\cos k_1\bar{x}-\operatorname{ch}k_1\bar{x})+C_2(\sin k_1x-\operatorname{sh}k_1\bar{x}). \tag{6.9}$$

[1]K. Gogenemzer and W. Prager. The Dynamics of Structures (Dinamika sooruzheniy). ONTI, 1936.

On the basis of (6.7) and (6.9) we obtain

$$\Delta(W,\ W')=-2D_1k_1/l_1,\qquad \Delta(W',\ M)=S_1k_1^3/l_1^3,$$
$$\Delta(WM)=2B_1k_1^2/l_1^2,\qquad \Delta(W'Q)=2A_1k_1^4/l_1^4,\tag{6.10}$$
$$\Delta(WQ)=S_1k_1^3/l_1^3,\qquad \Delta(MQ)=2E_1k_1^5/l_1^5.$$

In terms of the Prager functions we have the following for the second shell

$$W=C_1\left[\frac{C-S}{2}\cos k_2\bar{x}+A\sin k_2\bar{x}+\operatorname{ch}k_2\bar{x}\right]+$$
$$+C_2\left[-B\cos k_2x+\frac{C+S}{2}\sin k_2x+\operatorname{sh}k_2x\right].\tag{6.11}$$

On the basis of (6.11) the elementary determinants can be written in the form

$$\Delta(MQ)=2D_2\frac{k_2^5}{l_2^5},\quad \Delta(WQ)=-s_2k_2^3/l_2^3,$$
$$\Delta(W'Q)=2A_2\frac{k_2^4}{l_2^4},\quad \Delta(WM)=2B_2k_2^2/l_2^2,\tag{6.12}$$
$$\Delta(W'M)=-S_2\frac{k_2^3}{l_2^3},\quad \Delta(WW')\,2E_2k_2/l_2.$$

Substituting (6.10) and (6.12) into (6.5), we obtain the basic equation in the form /160

$$\nu^2\frac{k_2^4}{l_2^4}D_1D_2-\nu\left[\frac{k_1}{l_1}\frac{k_2^3}{l_2^3}B_1A_2+\frac{k_1^3}{l_1^3}\frac{k_2^3}{l_2^3}\frac{S_1S_2}{2}+\right.$$
$$\left.+\frac{k_1^3}{l_1^3}\frac{k_2}{l_2}A_1B_2\right]+\frac{k_1^4}{l_1^4}E_1E_2=0.\tag{6.13}$$

If the rigidity of the basic and of the supported shell is the same, i.e., $\nu_1=1\ \frac{k_1}{l_1}=\frac{k_2}{l_2}$, we obtain

$$(D_1D_2+E_1E_2)-(B_1A_2+A_2B_1)-\frac{S_1S_2}{2}=0.\tag{6.14}$$

Substituting the values of the function D_1, D_2, A_1, A_2, B_1, B_2, S_1, S_2, into (6.14), we obtain

$$1+\operatorname{ch}k_m\cos k_m=0.$$

In this case we obtain the same values for k_m as in section 3., i.e., for the composite shell the function $W_m^{(x)}$ is determined in the same way as for the continuous shell.

In all of the remaining cases equation (6.5) is used to determine k_m and to determine the approximating functions for each shell, which can be expressed in terms of one constant A_m by means of equation (6.4). The natural frequency of oscillations is determined from a system of n equations which has the form (7.1)

$$\frac{\partial}{\partial A_{mn}}[(P_1+P_2)-p^2(T_1+T_2)]=0,$$

where P_1 and P_2 are the potential energies of the first and second shell, respectively, T_1 and T_2 are the kinetic energies of the first and second shell, respectively, determined by means of equations (1.7) and (1.9). As an example we shall consider the results of testing the freely suspended, cylindrical shell reinforced at the ends.

The Geometric Characteristics of the Shell

The basic shell has the following parameters

$$l=232\ мм,\quad h=0.7\ мм,\quad R=117\ мм,$$

$$\frac{1}{2\Pi R}\sqrt{\frac{E}{0.91\varrho}}=7300,\quad \beta=\frac{h^2}{12R^2}=1.52\cdot 10^{-6},$$

while the supporting shell has the following parameters

$$\bar{l}=9\ мм,\ \beta_2=2.08\cdot 10^{-4}.$$

Since the flexural rigidity of the supporting shell is substantially greater than the rigidity of the basic shell $\beta_2 > \beta_1$, we can examine the spectrum of the composite shell approximately as the combination of the spectrum of a ring and a rigidly supported shell.

Figure 3 shows the theoretical curves and the experimental points. Qualitatively the picture is confirmed experimentally. In regard to the fact that the frequencies do not coincide, we may explain this by the circumstance that in this case we cannot limit ourselves to a single term of equation (6.5) and also by the fact that the true geometry of the shell differs from the one which has been assumed.

II. Conical Shell

Cylindrical, conical and toroidal shells represent shells of revolution. We select a system of coordinates in such a way that it coincides with the principal lines of curvature. /162

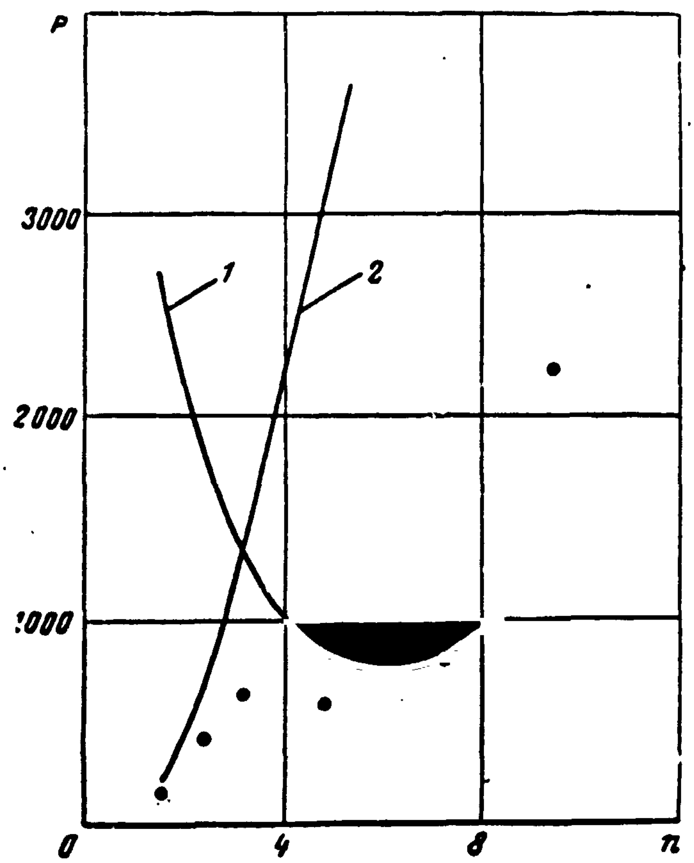

Figure 3. Theoretical and experimental frequencies of a reinforced shell. 1, Variations in natural frequencies of oscillations of ring; 2, variations in natural frequencies of rigidly supported shell.

1. Geometric Parameters and Strain Parameters

We direct the X axis along the generatrix of the cone (figure 1b); we designate by α the angle of conicity of the shell; we designate by R_0 and R_1 the respective radii of the small and large base. Obviously the radius of an arbitrary angular cross section will be equal to

$$r = R_0 + x \sin \alpha. \tag{1.1}$$

The position on the parallel ring is determined by the angle θ. Retaining the symbols which have been adopted in the theory of shells (ref. 2), we obtain the following expressions for the geometric parameters

$$a = x, \quad A = 1, \quad \beta = 0, \quad B = r; \quad \frac{1}{R_1} = \frac{1}{R_{12}} = 0; \quad \frac{1}{R_2} = \frac{\cos \alpha}{r}. \tag{1.2}$$

To obtain the strain parameters we represent the displacement along the normal to the middle surface by w, along the tangent to the circle of radius r by z and along the generatrix by u. Then we obtain the following equations for the strain parameters of elongation

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_\theta=\frac{1}{r}\left(\frac{\partial v}{\partial \theta}+u\sin\alpha-w\cos\alpha\right).$$
$$\omega=\frac{1}{r}\frac{\partial u}{\partial \theta}+r\frac{\partial}{\partial x}\left(\frac{v}{r}\right). \tag{1.3}$$

The parameters of flexural deformation are

$$\varkappa_1=\frac{\partial^2 w}{\partial x^2},$$
$$\varkappa_2=\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\cos\alpha\right)+\frac{\sin\alpha}{r}\frac{\partial w}{\partial x}, \tag{1.4}$$
$$\tau=\frac{1}{r}\left(\frac{\partial^2 w}{\partial x\partial \theta}+\frac{\partial v}{\partial x}\cos\alpha\right)-\left(\frac{\partial w}{\partial \theta}+v\cos\alpha\right)\frac{\sin\alpha}{r^2}.$$

2. Selection of the Approximating Functions

The natural oscillation functions w, v and u are selected as a sum of the products of two functions: one depending on x and the other depending on θ, specifically

$$w=\sum\sum A_{mn}W_m(x)\cos n\theta,$$
$$v=\sum\sum B_{mn}V_m(x)\sin n\theta; \tag{2.1}$$
$$u=\sum\sum C_{mn}U_m(x)\cos n\theta,$$

A_{mn}, b_{mn}, C_{mn} are arbitrary parameters.

In selecting the approximating functions we follow the method adopted and verified in the design of the cylindrical shell. Specifically we assume that $W_m(x)$ is a beam function, $X_m(x)$, $V_m(x)$ and $U_m(x)$ as well as the parameters B_{mn} and C_{mn} are not independent quantities, but are associated with $W_m(x)$ and A_{mn} by means of auxiliary conditions. These conditions are reduced to a situation where we assume that $\varepsilon_\theta = \omega = 0$.

Substituting (2.1) into (1.3) we find that for arbitrary values of θ and φ the following conditions must be satisfied /163

$$B_{mn}V_m(x)n+C_{mn}U_m(x)\sin\alpha-A_{mn}X_m\cos\alpha=0,$$
$$B_{mn}\left[V'_m(x)-\frac{V_m(x)}{r}\sin\alpha\right]-nC_{mn}\frac{U_m(x)}{r}=0. \tag{2.3}$$

We obtain the following expression from the first equation, which is accurate to within the longitudinal component,

$$B_{mn}V_m(x)=\frac{A_{mn}}{n}X_m\cos\alpha. \qquad (2.4)$$

Substituting the value of $B_{mn}\ V_m(x)$ into the second equation, we find (2.3)

$$C_{mn}U_m(x)=\frac{A_{mn}}{n^2}r\cos\alpha\left(X'_m(x)-\frac{X_m(x)}{r}\sin\alpha\right). \qquad (2.5)$$

Thus, on the basis of (2.4) and (2.5) and the assumption that $W_m(x) = X_m(x)$, we obtain the following expression for the approximating functions in place of (2.1)

$$\begin{aligned} w&=\sum\sum A_{mn}X_m\cos n\theta,\\ v&=\sum\sum\frac{A_{mn}}{n}X_m\cos\alpha\sin n\theta;\\ u&=\sum\sum\frac{A_{mn}}{n^2}(X'_m r-X_m\sin\alpha)\cos\alpha\cos n\theta.\end{aligned} \qquad (2.6)$$

3. Determination of those Strain Parameters Related to Selected Approximating Functions

On the basis of (2.1) the expression for the tensional strain parameters may be written as

$$\left.\begin{aligned} \varepsilon_1&=\sum\sum\frac{A_{mn}}{n^2}X^{\cdot\cdot}_m r\cos\alpha\cos n\theta,\\ \varepsilon_2&=\sum\sum\frac{A_{mn}}{n^2}\left(X'_m-\frac{X_m}{r}\sin\alpha\right)\frac{\sin 2\alpha}{2}\cos n\theta.\end{aligned}\right\} \qquad (3.1)$$

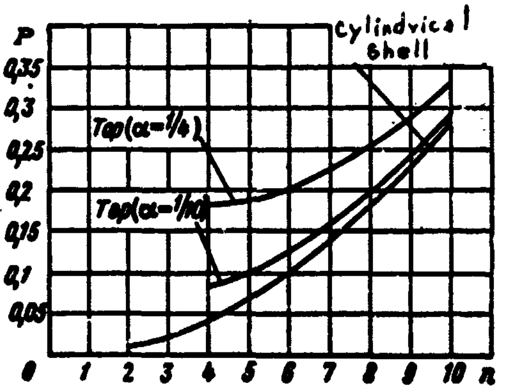

Figure 4. Comparison of relative frequencies p* of toroidal shells (α = 1/2, α = 1/10, h/r = 0.01) with those of cylindrical shell (r/l = 1/8 π, h/r = 0.01) for m = 1 and various n.

It follows from expression (3.1) that for this selection of the functions ϵ_2 becomes equal to 0 only for shells of small conicity or for large values of n.

The parameters of flexural strain will be

$$
\begin{aligned}
\varkappa_1 &= \sum\sum A_{mn} X''_m \cos n\theta, \\
\varkappa_2 &= \sum\sum A_{mn}\left[\frac{X_m}{r^2}(\cos^2\alpha - n^2) + \frac{X'_m}{r}\sin\alpha\right]\cos n\theta, \\
\tau &= \sum\sum A_{mn}\left(\frac{X'_m}{r} - \frac{X_m \sin\alpha}{r^2}\right)\frac{\cos^2\alpha - n^2}{n}\sin n\theta.
\end{aligned}
\tag{3.2}
$$

4. Computation of the Coefficients in the Equations

We compute the frequencies of the natural oscillations of a conical shell by the Ritz method. For this purpose we solve a system of equations of the form (1.10). The diagonal terms of the equations are written in the following manner /164

$$
\begin{aligned}
a_{mn}^{mn} = {} & \frac{Eh\pi}{1-\mu^2}\frac{\cos^2\alpha}{n^4}\left\{\int_0^l \left[X''^2_m r^2 + \sin^2\alpha\left(X'_m - \frac{X_m}{r}\sin\alpha\right)^2 + \right.\right. \\
& \left.\left. + 2\mu\sin\alpha X''_m r\left(X'_m - \frac{X_m}{r}\sin\alpha\right)\right] r\,dx\right\} + \frac{Eh^3}{12(1-\mu^2)}\pi\left(\int_0^l \left\{X''^2_m + \right.\right. \\
& + \left[\frac{X_m}{r^2}(\cos^2\alpha - n^2) + \frac{X'_m}{r}\sin\alpha\right]^2 + 2\mu X''_m\left[\frac{X_m}{r^2}(\cos^2\alpha - n^2) + \right. \\
& \left.\left.\left. + \frac{X'_m}{r}\sin\alpha\right] + 2(1-\mu)\left(\frac{\cos^2\alpha - n^2}{n}\right)^2\left(\frac{X'_m}{r} - \frac{X_m \sin\alpha}{r^2}\right)^2\right\} r\,dx\right),
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
b_{mn}^{mn} = \varrho h\frac{\pi}{n^4}\int_0^l & \left[(n^4 + n^2\cos^2\alpha) X_m^2 + \right. \\
& \left. + (X'_m r - X_m \sin\alpha)^2\cos^2\alpha\right] r\,dx,
\end{aligned}
\tag{4.2}
$$

The supplementary coefficients of equations (1.10) usually do not become equal to zero, and the functions x_m are selected in such a way that the condition $\int_0^l X_m X_p dx = 0\ (m \neq p)$, rather than the condition $\int_0^l X_m X_p r\,dx = 0$ is satisfied.

Consequently we obtain

$$a_{mn}^{pn}=\frac{Eh\pi}{1-\mu^2}\frac{\cos^2\alpha}{n^4}\int\limits_0^l\Big\{X''_mX''_pr^2+\sin^2\alpha\left(X'_m-\frac{X_m}{r}\sin\alpha\right)\times$$

$$\times\left(X''_p-\frac{X_p}{r}\sin\alpha\right)+\mu\left[X''_m\left(X'_p-\frac{X_p}{r}\sin\alpha\right)+\right.$$

$$\left.+X''_p\left(X'_m-\frac{X_m}{r}\sin\alpha\right)\right]r\Big\}r\,dx+\frac{Eh^3\pi}{12(1-\mu^2)}\times$$

$$\times\int\limits_0^l\Big\{X''_mX''_p+\left[\frac{X_m}{r^2}(\cos^2\alpha-n^2)+\frac{X'_m}{r}\sin\alpha\right]\times$$

$$\times\left[\frac{X_p}{r^2}(\cos^2\alpha-n^2)+\frac{X'_p}{r}\sin\alpha\right]+ \tag{4.3}$$

$$+\mu X''_m\left[\frac{X_p}{r^2}(\cos^2\alpha-n^2)+\frac{X'_p}{r}\sin\alpha\right]+\mu X''_p\times$$

$$\times\left[\frac{X_m}{r^2}(\cos^2\alpha-n^2)+\frac{X'_m}{r}\sin\alpha\right]+2(1-\mu)\frac{(\cos^2\alpha-n^2)^2}{n^2}\times$$

$$\times\left(\frac{X'_m}{r}-\frac{X_m\sin\alpha}{r^2}\right)\left(\frac{X'_p}{r}-\frac{X_p\sin\alpha}{r^2}\right)\Big\}r\,dx,$$

$$b_{mn}^{pn}=\varrho h\frac{\pi}{n^4}\int\limits_0^l[(n^4+n^2\cos^2\alpha)X_mX_p+\cos^2\alpha\times$$

$$\times(X'_mr-X_m\sin\alpha)(X'_pr-X_p\sin\alpha)]\,r\,dx. \tag{4.4}$$

5. A Shell With Hinged Edges

Let us consider a simple computational example when the edges of the shell are hinged. In this case we assume that /165

$$X_m=\sin\frac{m\pi x}{l}=\sin k_mx, \tag{5.1}$$

In this case it is obvious that the necessary boundary conditions are satisfied at the edges of the shell, i.e.,

$$w=0,\ M=0. \tag{5.2}$$

(the second condition (5.2) with an accuracy up to $\mu\frac{X_m}{r}\sin\alpha$

Before beginning the computation of coefficients (4.1)-(4.4), we introduce the following designations

$$R_{\text{cp}}=\frac{R_1+R_0}{2},\quad \lambda=\frac{R_1-R_0}{R_0}. \tag{5.3}$$

Part of the integrals is not obtained in a closed form. Therefore, by expanding the function under the integral sign in a series and integrating term by term, we obtain the required values of the integrals. To simplify we designate

$$\left.\begin{aligned} &\int_0^l \frac{X_m^2}{r}\,dx = \frac{l}{2R_0} L_0, \\ &\int_0^l \frac{\cos 2k_m x\,dx}{r} = \frac{l}{R_0} L_1, \\ &\int_0^l \frac{X_m^2}{r^3}\,dx = \frac{l}{2R_0^3} L_2, \end{aligned}\right\} \tag{5.4}$$

where

$$L_0 = 1 + \sum_i (-\lambda)^i q_i, \tag{5.5}$$

$$L_1 = -\sum_i \lambda^i \left(\frac{1}{i+1} - q_i\right), \tag{5.6}$$

$$L_2 = 1 + p_0\lambda + \sum_{i=1}^{n} \varrho_i \lambda^{i+1}, \tag{5.7}$$

$$q_i = \sum_n \frac{(2m\pi)^{2n}}{2n!} \frac{(-1)^{n+i}}{2n+i+2}, \tag{5.8}$$

$$p_0 = \sum_n (-1)^{n+1} \frac{(2m\pi)^{2n}}{2n!} \left[\frac{n-1}{1+\lambda} - \frac{1}{2(1+\lambda)^2} - \frac{n(2n-1)}{2n+2}\right], \tag{5.9}$$

$$p_i = \sum_n (-1)^{n+1} \frac{(2m\pi)^{2n}}{2n!} \frac{n(2n-1)}{2n+i+2}, \tag{5.10}$$

where m is the number of waves in the shell in the longitudinal direction. /166
From (5.4)-(5.10) and the table of elementary integrals we obtain

$$\begin{aligned} n^4 \frac{R_0 R_{cp}(1-\mu^2)}{l R_{cp} E} a_{mn}^{mn} = {} & \cos^2\alpha \Big[k_m^4 R_0^3 R_{cp} \frac{(1+\lambda)^2+1}{2} + \\ & + 0{,}7 k^2 R_0 R_{cp} \sin^2\alpha + L_0 \sin^4\alpha\Big] + \beta n^4 \{k^4 R_0^3 R_{cp} + L_2 (\cos^2\alpha - \\ & - n^2)^2 \left(1 + \frac{2(1-\mu^2)}{n^2}\sin^2\alpha\right) + k^2 R_0^2 \Big[\Big[\frac{\ln(1+\lambda)}{\beta}[\sin^2\alpha + \\ & + 2(1-\mu)\left(\frac{\cos^2\alpha - n^2}{n}\right)^2 - 2\mu(\cos^2\alpha - n^2)\Big] + L_1[\sin^2\alpha + \\ & + 2(1+\mu)(\cos^2\alpha - n^2) - 2(1-\mu)\left(\frac{\cos^2\alpha - n^2}{n}\right)^2\Big]\Big], \end{aligned} \tag{5.11}$$

$$\frac{n^4}{l R_{cp}} b_{mn}^{mn} = \left\{n^4 + \cos^2\alpha\left[n^2 + \frac{9}{2}\sin^2\alpha + k_n^2 R_0^2 \frac{(1+\lambda)^2+1}{2}\right]\right\}. \tag{5.12}$$

We neglect the coefficients a^{mn}_{pn} and b^{mn}_{pn}. In the first approximation the frequency of the natural oscillations of a conical shell may be determined by setting the diagonal terms equal to zero, i.e.,

$$p^2_{1m} = \frac{a^{mn}_{mn}}{b^{mn}_{mn}}. \tag{5.13}$$

For large values of n and for a large conicity, this equation will not be valid, because in a conical shell there is always a coupling between the oscillation modes w_m and w_{m+1}.

Therefore, it is necessary to compute p_1^2 from the determinant of the following form:

$$\begin{vmatrix} a^{mn}_{m,n} - p_1^2 b^{mn}_{mn} & a^{m+1,n}_{m,n} - p_1^2 b^{m+1,n}_{mn} \\ a^{m+1,n}_{m,n} - p_1^2 b^{m+1,n}_{m,n} & a^{m+1,n}_{m+1,n} - p_1^2 b^{m+1,n}_{m+1,n} \end{vmatrix} = 0, \tag{5.14}$$

where the expression for size and diagonal coefficients is given by equations (4.1)-(4.4).

6. Numerical Example

Ye. Breslavskiy tested and computed two conical shells which were hinge-supported at the ends.

Their geometric characteristics are shown in table 5.

We compute the frequencies for the case m = 1, n = 2, 4, 6 and 8, i.e., for a single longitudinal semiwave and for 2, 4, 6 and 8 waves in the circumferential direction.

We carry out the computation by means of equation (5.13). From equation (5.4) we compute the coefficients associated with the conicity of the shell: L_0, L_1, L_2.

For the shells under consideration, angle α is small. Therefore /167
we can neglect the quantities $\sin^2 \alpha$ and $\sin^4 \alpha$ in expressions a^{mn}_{mn} and b^{mn}_{mn} and let $\cos^2 \alpha = 1$. Then the approximate equation for p^2 will be

$$p_1^2 = \{k_m \lambda^4 R_0^3 R_{cp}(1+\lambda+\lambda^2/2) + \beta n^4 [\lambda^4 R_0^3 R_{cp} + L_2(1-n^2)^2 + 2k_m \lambda^2 R_0^2 (1-n^2)] \left(\frac{\ln(1+\lambda)}{\lambda}\left(\frac{0,7}{n^2-1}\right) + L_1\left(2 - \frac{0,7}{n^2}\right)\right)\} : [n^4 + n^2 + k_m \lambda^2 R_0^2 (1+\lambda+\lambda^2/2)],$$

TABLE 5. GEOMETRIC CHARACTERISTICS OF THE SHELL.

№ of shell	R_0	R_1	ΔR	l	$\sin\alpha$	$\cos\alpha$	h	$\frac{n^2}{12R_0^2}=\beta$
1	10	17.5	7,5	60	0,125	0,992	$1\cdot10^{-2}$	$8,35\cdot10^{-6}$
2	12,5	15	2,5	85	0,0294	0,999	$1\cdot10^{-2}$	$5,36\cdot10^{-6}$

continued

№ of shell	R_{cp}	λ	$\sqrt{R_0R_{cp}}$	$\sqrt{\frac{E}{\varrho(1-\mu^2)}}$	$\frac{(13)}{2\pi(12)}$	$\left(\frac{\pi}{l}R_0\right)^2$	$\ln(1+\lambda)$
1	13,75	0,75	11.7	$5,3\cdot10^5$	$0,72\cdot10^4$	0,275	0,5591
2	13,75	0.2	13,10	$5,3\cdot10^5$	$0.641\cdot10^4$	0,211	0,198

cp = ave

where

$$\beta=\frac{h^3}{12R_0^2},$$

and the frequency of natural oscillations in Hz will be

$$p=\frac{p_1}{2\pi\sqrt{R_0R_{cp}}}\sqrt{\frac{E}{1-\mu^2}}.$$

For purposes of comparison table 6 presents the computed and experimental data.

In addition to our computed quantities p_1 and p_2, which are based /168
on the assumption that the transverse cross section is not elongated and that shear is absent in the middle surface for the cone, we present the computational data of Breslavskiy which take the deformation of the middle surface completely into account.

It follows from table 6 that as the conicity increases, the natural frequency of oscillations of the shells also increases. As far as the accuracy of our and Breslavskiy theoretical methods is concerned, it should be assumed that they are the same.

The advantage of our methods is reduced to the possibility of obtaining simple design equations. As the value of n increases, it obvious that the natural oscillations must be determined from equation (5.14).

TABLE 6.

	Shell № 1 $\alpha=7°10'$			Shell № 2 $\alpha=1°40'$		
n	$p_э$	theoretical		$p_э$	theoretical	
		p_B	cone p		p_B	cone p
2	—	—		276	311	332
3	310	326	360	200	192	192
4	285	254	270	244	219	222
6	456	447	455	410	454	450

$p_э$ = experimental frequencies

p_B = Breslavskiy data

III. Toroidal Shell

A shell of this type is formed by the rotation of a ring (figure 1c) of radius r with respect to the z axis. We designate the distance between the center of the ring and the axis by R_0. Since the literature contains no data, either theoretical or experimental, on the frequency spectrum of natural oscillations, in practical calculations the torus is sometimes replaced by a cylindrical shell, whose length l is equal to the average perimeter of the torus, radius r is equal to the smaller radius of the torus. While the torus represents a closed surface, and therefore the rigidity of such a shell substantially exceeds the rigidity of a cylindrical shell. The high rigidity of a torus compared with that of the cylinder has its effect not only on the absolute value of the frequencies, but also on the nature of their distribution. In other words, if the lower frequency of natural oscillations of a cylindrical shell is associated with a minimum number of waves in the longitudinal direction, then this condition is not observed for the torus. The lower frequency is associated not only with the geometric characteristics, but also with a certain number of waves in directions θ and φ.

1. Geometric Parameters and Strain Parameters

The radius of an arbitrary angular cross section will be

$$R=R_0+r\cos\varphi=R_0(1+\alpha\cos\varphi). \tag{1.1}$$

The position of the point on the surface of the torus is determined by a system of orthogonal curvilinear coordinates θ and φ.

Using the symbols presented in reference 2, the geometric parameters of the torus will be

$$\alpha=\theta;\quad A=R_0(1+\alpha\cos\varphi);\quad \beta=\varphi;\quad B=r;$$
$$\frac{1}{R_1}=\frac{\cos\varphi}{A};\quad \frac{1}{R_2}=\frac{1}{r};\quad \frac{1}{R_{12}}=0. \tag{1.2}$$

To obtain the strain parameters we let
w be the displacement along the normal to the middle surface;
v be the displacement along the tangential coordinate θ = const.
u be the displacement along the tangential coordinate φ = const.

From the general equations of the theory of shells the strain parameters for elongation will be

$$\varepsilon_\theta=\frac{1}{A}\left(\frac{\partial u}{\partial\theta}-v\sin\varphi-w\cos\varphi\right);$$
$$\varepsilon_\varphi=\frac{1}{r}\left(\frac{\partial v}{\partial\varphi}-w\right);\quad \omega=\frac{A}{r}\frac{\partial}{\partial\varphi}\left(\frac{u}{A}\right)+\frac{1}{A}\frac{\partial v}{\partial\theta}. \tag{1.3}$$

The flexural parameters will be as follows /169

$$\varkappa_1=\frac{1}{A}\left[\frac{1}{A}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta}+u\cos\varphi\right)-\frac{\sin\varphi}{r}\left(\frac{\partial w}{\partial\varphi}+v\right)\right],$$
$$\varkappa_2=\frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial w}{\partial\varphi}+v\right), \tag{1.4}$$
$$\tau=\frac{1}{rA}\alpha\left[\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\varphi}+v\right)+\frac{\sin\varphi}{1+\alpha\cos\varphi}\left(\frac{\partial w}{\partial\theta}+u\cos\varphi\right)+\right.$$
$$\left.+\frac{\partial u}{\partial\varphi}\cos\varphi\right].$$

Equations are real not only for a closed torus but for any toroidal surface.

2. Selection of the Approximating Functions

In determining the frequencies of natural oscillations of a toroidal shell as well as in computing cylindrical and conical shells we utilize the Ritz method. The success of this method is determined to a large extent by the selection of the approximation functions. It is expedient to determine the displacements in the form of a double series of fucntions, i.e.,

$$w=\sum\sum L_{mn}W_m(\theta)W_n(\varphi),$$
$$v=\sum\sum M_{mn}V_m(\theta)V_n(\varphi), \tag{2.1}$$
$$u=\sum\sum N_{mn}U_m(\theta)U_n(\varphi).$$

Let us assume that $W_m(\theta) = \cos m\theta$ determines the oscillations of an elementary ring of radius R; $W_n(\varphi) = \sin n\varphi$ determines the oscillations of an elementary ring of radius r; m and n are whole numbers. Imposing the additional conditions $\varepsilon_\varphi = \omega = 0$ on the displacements u, v and w, we obtain

$$\begin{aligned} w &= \sum\sum L_{mn} \cos m\theta \sin n\varphi, \\ v &= \sum\sum L_{mn} \cos m\theta \frac{\cos n\varphi}{n}, \\ u &= -\alpha(1+\alpha\cos\varphi)\sum\sum L_{mn} \sin m\theta \frac{m}{n^2} U_n^*, \end{aligned} \tag{2.2}$$

where

$$U_n^* = n\int \cos n\varphi d\varphi/(1+\alpha\cos\varphi)^2.$$

We determine the strain parameters for the selected functions

$$\begin{aligned} \varepsilon_\theta &= \sum\sum L_{mn} \frac{\cos m\theta}{R}\left[\frac{m^2}{n^2} U_n^* \alpha + \right. \\ &\left. + \frac{(n-1)\sin(n+1)\varphi + (n+1)\sin(n-1)\varphi}{2n(1+\alpha\cos\varphi)}\right], \\ \varepsilon_\varphi &= \omega = 0, \end{aligned} \tag{2.3}$$

$$\left.\begin{aligned} \varkappa_1 &= -\sum\sum \frac{L_{mn}\cos m\theta}{R_0 r(1+\alpha\cos\varphi)^2}\left[\alpha^2 \frac{m^2}{n^2} U_n^* \cos\varphi + \right. \\ &\left. + \frac{n^2-1}{n}\cos n\varphi\cos\varphi + \alpha m^2 \frac{\sin n\varphi}{1+\alpha\cos\varphi}\right], \\ \varkappa_2 &= -\sum\sum L_{mn} \frac{\cos m\theta}{r^2}(n^2-1)\sin n\varphi, \\ \tau &= -\sum\sum L_{mn} \frac{\sin m\theta}{Rr(1+\alpha\cos\varphi)} m\left[\frac{n^2-1}{n}\cos n\varphi + \right. \\ &\left. + \alpha\frac{(1+n^2)\cos(n-1)\varphi + (1-n^2)\cos(n+1)\varphi}{2n^2(1+\alpha\cos\varphi)}\right]. \end{aligned}\right\} \tag{2.4}$$

3. The Formulation of the Characteristic Equation

Having determined the strain parameters and having formulated the expressions for the potential P and kinetic energy T, we differentiate P and T with respect to L_{mn} and find the coefficients of the characteristic equations. Since the system of functions which depend on θ is orthogonal, we obtain a system of the following form rather than a single characteristic determinant /170

$$\begin{vmatrix} d^{mn}_{mn} - x^2 b^{mn}_{mn} & d^{mn}_{mn+1} - x^2 b^{mn}_{mn+1} \cdots \\ d^{mn+1}_{mn} - x^2 b^{mn+1}_{mn} & d^{mn+1}_{mn+1} - x^2 b^{mn+1}_{mn+1} \cdots \\ \cdots & \cdots \end{vmatrix} = 0, \tag{3.1}$$

where $d^{mn}_{mn} = a^2 a^{mn}_{mn}\ (I) + \beta a^{mn}_{mn}\ (II)$,

$$\beta = \frac{h^2}{12r^2}, \quad x^2 = p^2 r^2 \frac{1-\mu^2}{E} \varrho. \tag{3.2}$$

After simple algebraic transformations, the coefficients a^{mn}_{mn} (I), a^{mn}_{mn} (II), b^{mn}_{mn} may be written in the following manner

$$n^4 b^{mn}_{mn} = n^4 + n^2 + \alpha^2 \frac{m^2}{\pi^2} \int_0^{2\pi} U_n^{*2}(\varphi)(1+\alpha\cos\varphi)^3 d\varphi. \tag{3.3}$$

The last integral is also taken in series for small α and large n ($n > 4$) (this term may be neglected)

$$n^4 a^{mn}_{mn}(I) = \int_0^{2\pi} \left[m^2 U_n^* \alpha + \frac{n}{2} \frac{(n-1)\sin(n+1)\varphi + (n+1)\sin(n-1)\varphi}{1+\alpha\cos\varphi} \right]^2 \times$$
$$\times (1+\alpha\cos\varphi)\, d\varphi,$$

$$n^4 a^{mn}_{mn}(II) = n^4 \int_0^{2\pi} \left\{ (n^2-1)^2 \sin^2 n\varphi + \frac{\alpha^2}{(1+\alpha\cos\varphi)^2} \times \right.$$
$$\times \left[\alpha^2 \frac{m^2}{n^2} U_n^* \cos\varphi + \frac{n^2-1}{n} \cos n\varphi \sin\varphi + \alpha m^2 \frac{\sin n\varphi}{1+\alpha\cos\varphi} \right]^2 + \tag{3.4}$$
$$+ 2\mu\alpha \frac{(n^2-1)\sin n\varphi}{1+\alpha\cos\varphi} \left[\alpha^2 \frac{m^2}{n^2} U_n^* \cos\varphi + \right.$$
$$\left. + \frac{n^2-1}{n} \cos n\varphi \sin\varphi + \alpha m^2 \frac{\sin n\varphi}{1+\alpha\cos\varphi} \right] +$$

$$+ 2(1-\mu) \frac{\alpha^2 m^2}{(1+\alpha\cos\varphi^2)} \left[\frac{n^2-1}{n} \cos n\varphi + \frac{\alpha}{2n^2} \times \right.$$
$$\left. \left. \times \frac{(1+n^2)\cos(n-1)\varphi + (1-n^2)\cos(n+1)\varphi}{1+\alpha\cos\varphi} \right] \right\} (1+\alpha\cos\varphi)\, d\varphi. \tag{3.5}$$

The integration of coefficients a^{mn}_{mn} is associated with substantial computational difficulty. The integrals are not determined in closed form and they must be computed by means of series. Since α is a small quantity, /171

$0.1 < \alpha < 0.5$ we may retain only the terms α^0 and α^2 in coefficients a_{mn}^{mn} (I) and a_{mn}^{mn} (II), specifically

$$n^4 a_{mn}^{mn}(I) \approx m^4 \alpha^2 (L_0 + L_1) + \alpha m^2 n^2 \frac{n^2+1}{n^2-1} A_1 + \\ + \frac{n^2}{2}\left[B_0(n^2+1) + B_2 \frac{n^2-1}{2}\right], \tag{3.6}$$

$$a_{mn}^{mn}(II) \approx (n^2-1)^2 + \alpha^2\left[\frac{n^2-1}{2n^2}\left(B_0 - \frac{B_2}{2}\right) + 2m^2(n^2-1) \times \\ \times B_0\left(\mu + (1-\mu)\frac{n^2-1}{n^2}\right)\right]. \tag{3.7}$$

However, we cannot neglect the quantity $\alpha^2 \alpha_{mn}^{mn}$ (I) compared with the quantity $\beta^* n^4$ in the diagonal terms, because the order of magnitude for the series of the values n and α is the same. In the side terms we retain only the quantities $\alpha^2 \alpha_{mn}^{mn+1}$.

The parameters L_0, L_1, A_1, B_0 and B_2 in equations (3.6) and (3.7) have the following expressions

$$L_0 = \int_0^{2\pi} U_n^{*2} d\varphi, \quad L_1 = \int_0^{2\pi} U_n^{*2} \cos\varphi \, d\varphi,$$

$$B_0 = \sum_{l=0} \left(\frac{\alpha}{2}\right)^{2l} \frac{2l!}{l!^2},$$

$$A_1 = -2\sum_{l=0}\left(\frac{\alpha}{2}\right)^{2l+1} \frac{(2l+2)!}{(l+1)!\,l!}, \tag{3.8}$$

$$B_2 = 2\sum_{l=0}\left(\frac{\alpha}{2}\right)^{2+2l} \frac{(2l+2)!}{(l+2)!\,l!}.$$

For convenience of computations we present the values of L_0 and L_1 in the table

α	n	5	6	8	9	10
1/2	L_0	3.17	3.11	3.06	3.05	3.04
	L_1	−3.20	−3.06	−2.96	−2.93	−2.93
1/3	L_0	1.60	1.59	1.58	1.58	1.58
	L_1	−1.18	−1.13	−1.09	−1.08	−1.08

To evaluate the frequencies of natural oscillations we utilize the following approximate equation

$$p^2 < \frac{E}{1-\mu^2}\frac{1}{\varrho r^2}\frac{\alpha^2 a_{mn}^{mn}(I) + \beta n^4 a_{mn}^{mn}(II)}{n^4+n^2}, \tag{3.9}$$

which has the following expanded form /172

$$p^2 < \frac{E}{1-\mu^2}\frac{1}{\varrho r^2}\left[B_0\frac{\alpha^2}{2} + \frac{\beta^2 n^4(n^2-1)+\alpha^4 m^4(L_0+\alpha L_1)+\alpha^3 m^2 n^2 A_1}{n^2(n^2+1)}\right]. \tag{3.10}$$

For small values of α the frequency of natural oscillations is practically independent of m, which gives the number of waves along the radius of the large circle.

For small values of α, i.e., if $\alpha^2/2 \leq \beta n^4$, the frequency is computed by means of the equation for an infinitely long cylindrical shell. Figure 4 shows the variation in the relative frequencies of natural oscillations p^* of toroidal shells with parameters $\alpha=1/4$, $\alpha = 1/10$ and $h/r = 0.01$ and of a cylindrical shell for which $r/l = \frac{1}{8\pi}$, $h/r = 0.01$ (the quantities $p\sqrt{\frac{1-\mu^2}{E}\rho}$) are compared). As α decreases in value, the frequencies of the cylindrical and toroidal shells will approach each other. For small values of n the difference in the frequencies will be substantial due to the difference in the coefficients $a_{mn}^{mn}(I)$. It is known that the principal role in the flexural oscillations of thin-walled shells is taken into account as the tensional energy, as well as the flexural energy. This differs from the case of the cylinder where the tensional energy is determined only by the longitudinal displacement u. The equations for the cylindrical shell may be used only when n > 10.
played by the displacement w, which for a torus is

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